前回の再整理

d 次元立方格子上の対称ランダムウォーク. 各方向へのジャンプ確率 $1/2d$. $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{R}^d$.

生成関数を次で定義.

$$Z(s_1, \ldots, s_d, t) := \sum_{\mathbf{x}} s_1^{x_1} \cdots s_d^{x_d} P(\mathbf{x}, t). \tag{4}$$

本質的に生成関数と同じものを次で定義.

$$\tilde{P}(\mathbf{k}, t) := \frac{1}{(2\pi)^{d/2}} Z(\mathrm{e}^{-ik_1}, \ldots, \mathrm{e}^{-ik_d}, t) = \frac{1}{(2\pi)^{d/2}} \sum_{\mathbf{x}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} P(\mathbf{x}, t) \tag{5}$$

$\tilde{P}$ は P の Fourier 級数変換. だから, 逆に,

$$P(\mathbf{x}, t) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d/2}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \tilde{P}(\mathbf{k}, t). \tag{6}$$

$d = 2$ の結果

$$Z(s, r, t) = \left[\frac{1}{4}(s + r + \tfrac{1}{s} + \tfrac{1}{r})\right]^t \tag{7}$$

の類推から,

$$Z(s_1, \ldots, s_d, t) = \left[\frac{1}{2d}(s_1 + \cdots + s_d + \tfrac{1}{s_1} + \cdots + \tfrac{1}{s_d})\right]^t, \tag{8}$$

$$\tilde{P}(\mathbf{k}, t) = \frac{1}{(2\pi)^{d/2}} \left[\frac{1}{d} \sum_{\mu=1}^{d} \cos k_\mu\right]^t. \tag{9}$$

t についての生成関数を次で定義.

$$G(\mathbf{x}, \lambda) := \lambda \sum_{t=0}^{\infty} \lambda^t P(\mathbf{x}, t) \tag{10}$$

$\tilde{P}$ で書き直して,

$$\begin{aligned} G(\mathbf{x}, \lambda) =& \lambda \int_{-\pi}^{\pi} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \sum_{t=0}^{\infty} \lambda^t \left[\frac{1}{d} \sum_{\mu=1}^{d} \cos k_\mu\right]^t \\ =& \lambda \int_{-\pi}^{\pi} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \, \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}}{1 - \frac{\lambda}{d}\sum_{\mu=1}^{d} \cos k_\mu}. \end{aligned} \tag{11}$$